Perfect harmony: A mathematical analysis of four historical tunings

Michael F. Page[a)]
The Pingry School, Martinsville Road, Martinsville, New Jersey 08836

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In Western music, a musical interval defined by the frequency ratio of two notes is generally considered consonant when the ratio is composed of small integers. Perfect harmony or an ''ideal just scale,'' which has no exact solution, would require the division of an octave into 12 notes, each of which would be used to create six other consonant intervals. The purpose of this study is to analyze four well-known historical tunings to evaluate how well each one approximates perfect harmony. The analysis consists of a general evaluation in which all consonant intervals are given equal weighting and a specific evaluation for three preludes from Bach's ''Well-Tempered Clavier,'' for which intervals are weighted in proportion to the duration of their occurrence. The four tunings, 5-limit just intonation, quarter-comma meantone temperament, well temperament (Werckmeister III), and equal temperament, are evaluated by measures of centrality, dispersion, distance, and dissonance. When all keys and consonant intervals are equally weighted, equal temperament demonstrates the strongest performance across a variety of measures, although it is not always the best tuning. Given C as the starting note for each tuning, equal temperament and well temperament perform strongly for the three ''Well-Tempered Clavier'' preludes examined. [DOI: 10.1121/1.1788732]

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I. INTRODUCTION

In Western music, a musical interval defined by the frequency ratio of two notes is generally considered pleasing or consonant when the ratio is composed of small integers. For example, an octave has a ratio of 2:1, and two notes an octave apart are often perceived as being essentially the same note. In addition to unison (1:1 ratio) and the octave (2:1), six other musical intervals are generally considered consonant: perfect fifth, perfect fourth, major and minor thirds, and major and minor sixths. As discussed by Tenney,[1] the meaning of consonance has evolved over time, but despite different theories about the causes of consonance, there is strong agreement that intervals defined by small integer ratios are the most consonant. Perfect harmony would require the division of an octave into 12 tones, each of which would be used as the root note for the other consonant intervals that have the following frequency ratios: 3:2 (perfect fifth), 4:3 (perfect fourth), 5:4 (major third), 6:5 (minor third), 5:3 (major sixth), and 8:5 (minor sixth). As described by Hall,[2] perfect harmony or an ''ideal just scale'' has no exact solution, that is, there is no set of 12 tones that can exactly meet all the constraints imposed by the small-integer ratios of the consonant intervals.

The purpose of this study is to analyze four well-known historical tunings, each of which defines a 12-tone scale, to evaluate how well each one approximates perfect harmony. Because each tuning includes unison and an octave by definition, the evaluation focuses on the six other consonant intervals. The four tunings in this study are particular cases of just intonation (JI), quarter-comma meantone temperament (MT), well temperament (Werckmeister III) (WT), and equal temperament[3] (ET). These were selected because each has held a prominent place in the evolution of tunings in Western music. The analysis consists of two phases: a general evaluation in which all consonant intervals are given equal weighting and a specific evaluation for three preludes from Bach's ''Well-Tempered Clavier,''[4] for which intervals are weighted in proportion to the duration of their occurrence in the music. Recent work on tempered intervals has focused on the perception of consonance (see, for example, Vos and van Vianen[5]). The analysis reported here uses functions of physical variables, specifically frequencies and spectra, which are correlated with the perceptual variables of pitch and timbre, to compare tunings. Such quantitative measures can be used to complement perceptual studies about the consonance of a tuning. In several areas, comparisons to the results of perceptual studies have been included here. Previous work by Hall[6] examined the performance of a wide range of tunings with respect to specific pieces of music using a single measurement statistic, while Rasch[7] examined a variety of regular[8] tunings using three criteria, but without examination of any specific music. This study extends their work by applying a comprehensive set of evaluation criteria, including the ones used by Hall and Rasch, and by evaluating regular and irregular tunings in the context of both equal and composition-specific weightings. This analysis also extends the work by Barnes[9] on the ''Well-Tempered Clavier'' to the extent that all consonant intervals, not just the major thirds, are examined; however, this study is not intended to justify any specific temperament as the one Bach intended for the ''Well-Tempered Clavier.'' Finally, the analysis includes a quantitative measure of dissonance introduced by Sethares[10] which is based on the Plomp–Levelt curves[11] for perception

[a)]Electronic mail: michaelfpage@att.net

TABLE I. Definitions of the intervals in a 12-tone scale relative to the tonic for four historical tunings. Columns (a) and (b) provide the common names and symbols for the intervals. Columns (c)–(f) provide the frequency ratios that define these intervals for one version of 5-limit just intonation, quarter-comma meantone temperament, well temperament (Werckmeister III), and equal temperament, respectively.

Interval name (a)	Interval symbol (b)	5-Limit just intonation (JI) (c)	Quarter-comma meantone temperament (MT) (d)	Well temperament (Werckmeister III) (WT) (e)	Equal temperament (ET) (f)
Unison	p1	1/1	1/1	1/1	1/1
Minor second	m2	16/15	$2187/2048\times(80/81)^{7/4}$	256/243	$2^{1/12}/1$
Major second	M2	9/8	$9/8\times(80/81)^{2/4}$	$9/8\times 1/P^{2/4}$	$2^{2/12}/1$
Minor third	m3	6/5	$32/27\times(81/80)^{3/4}$	32/27	$2^{3/12}/1$
Major third	M3	5/4	$81/64\times(80/81)^{4/4}=5/4$	$81/64\times 1/P^{3/4}$	$2^{4/12}/1$
Perfect fourth	p4	4/3	$4/3\times(81/80)^{1/4}$	4/3	$2^{5/12}/1$
Tritone	tt	45/32	$729/512\times(80/81)^{6/4}$	1024/729	$2^{6/12}/1$
Perfect fifth	p5	3/2	$3/2\times(80/81)^{1/4}$	$3/2\times 1/P^{1/4}$	$2^{7/12}/1$
Minor sixth	m6	8/5	$6561/4096\times(80/81)^{8/4}=25/16$	128/81	$2^{8/12}/1$
Major sixth	M6	5/3	$27/16\times(80/81)^{3/4}$	$27/16\times 1/P^{3/4}$	$2^{9/12}/1$
Minor seventh	m7	9/5	$16/9\times(81/80)^{2/4}$	16/9	$2^{10/12}/1$
Major seventh	M7	15/8	$243/128\times(80/81)^{5/4}$	$243/128\times 1/P^{3/4}$	$2^{11/12}/1$
Octave	p8	2/1	2/1	2/1 where P= 531441/ 524288	2/1

of pure tones, but which he generalized to account for the harmonic partials of complex tones.

Equal temperament, which divides an octave into 12 equally spaced half-tones, has been the standard tuning since the 19th Century. However, during the 20th Century, it has been criticized by proponents of just intonation as a ‘‘matter of expediency’’ since it simplifies instrument design and playing techniques (see, for example, Doty[12]). This study demonstrates that equal temperament performs strongly in approximating perfect harmony when all keys and all consonant intervals are given equal weighting compared to the particular cases of the other tunings examined, as well as for the performance of the three ‘‘Well-Tempered Clavier’’ preludes examined. The findings here agree with the results of Rasch[13] in demonstrating that equal temperament is an appropriate tuning for the ‘‘Well-Tempered Clavier.’’ Additionally, well temperament also performs strongly, especially for the specific ‘‘Well-Tempered Clavier’’ preludes. The results for the specific preludes are subject to the selection of C as the starting note for each tuning,

II. FOUR HISTORICAL TUNINGS INCLUDED IN THIS STUDY

A. 5-Limit just intonation

As defined by Barbour,[14] just intonation refers to any 12-note scale that contains some pure fifths and major thirds with the ratios given earlier. Probably the earliest example of just intonation is the tuning credited to the Greek mathematician and philosopher Pythagoras (6th Century B.C.). Pythagorean tuning is based on multiples of the prime numbers two and three, but because it does not utilize the number five, the thirds and sixths do not sound consonant by most standards. This tuning was popular, nonetheless, up through the Middle Ages.

From the 14th to the 16th Centuries, several music theorists were involved in introducing the use of consonant thirds and, as such, were forced to abandon Pythagorean tuning. Many variations of just intonation with some pure fifths and major thirds have been defined, with one common definition in terms of frequency ratios shown in Table I, column (c).[15] This particular version of just intonation agrees with what Doty[12] referred to as the ‘‘harmonic duodene of C.’’ The modifier ‘‘5-limit’’ indicates that the number five is the largest prime number used as a base number in the frequency ratios.

Although just intonation was widely accepted by theorists during the Renaissance, the growth of fixed-pitch instruments, such as pipe organs and keyboard instruments, meant that just intonation had limited use in practice. Modulation to other keys produces some very unacceptable intervals. Although the frequency ratios in Table I, column (c), for the consonant intervals are based on small integers, modulation to other keys creates ratios as complex as 1024/675.

Efforts were made during the 16th and 17th Centuries to create keyboard instruments with significantly more than 12 notes per octave in an attempt to achieve the ratios in Table I, column (c), for at least several keys. But these instruments did not become popular, and the concept of temperament, or altering the tuning of certain notes, took hold. For this study, an initial tone of C was set at 261.63 Hz and the ratios in Table I, column (c), were applied to obtain the remaining notes in a scale. These tones were modulated through all keys and analyzed to determine how close the 72 ratios formed by the six consonant intervals of interest are to the pure ratios for perfect harmony.[16]

B. Quarter-comma meantone temperament

Despite the interest of theorists in just intonation, meantone temperament, first introduced in the early 1400s, became the predominant method of tuning during the 16th and 17th Centuries. In meantone, the major thirds of Pythagorean tuning are made more consonant by a slight tempering of the perfect fifths. In addition, the tone located between the root note (say C) and the higher note in the major third (E) is positioned at the mean of the interval. This results in two tones (C–D and D–E) of equal size, whereas just intonation has tones of different sizes.

There are many ways to accomplish meantone, but the predominant method, quarter-comma meantone, was devised by Pietro Aaron in 1523. Each perfect fifth is reduced by one-quarter of a syntonic comma.[17] Then other tones are constructed by adding fifths together (multiplying their respective ratios), which after four fifths yields a major third. This process is continued until all but one of the fifths is tempered. The result is eight pure major thirds, but also one wide fifth, known as a ''wolf'' interval because of its sharp sound. The frequency ratios used to construct a 12-tone scale in quarter-comma meantone are shown in Table I, column (d). The complete set of frequency ratios for quarter-comma meantone temperament with each note used in turn as the root note was also calculated for comparison to the frequency ratios for perfect harmony.

C. Well temperament (Werckmeister III)

As composers started to make greater use of modulation in a single composition, the existence of wolf intervals in quarter-comma meantone made this tuning less attractive. This led to the development of well temperaments, including the popular Werckmeister III (c. 1691) named after its inventor, Andreas Werckmeister. In this temperament, the goal is to keep the fifths and thirds as close to just intonation as possible, while spreading out the wolf intervals. This is accomplished by spreading the Pythagorean comma,[18] the difference between twelve perfect fifths and seven octaves, across four intervals. As a result, the wolf intervals are eliminated, the major thirds are sharper, four of the perfect fifths have been flattened, and there are once again different sized tones.

In well temperament, all the keys are usable, but they sound different due to the different size of the tones. Different keys are thought to correspond to different ''colors'' or moods. The frequency ratios used to construct a 12-tone scale in well temperament (Werckmeister III) are shown in Table I, column (e). The complete set of frequency ratios with each note used in turn as the root note was also found.

Barnes[9] analyzed the suitability of Werckmeister III as the temperament intended by Bach for the ''Well-Tempered Clavier,'' concluding that the piece was written for a temperament similar to that tuning and that such temperaments are suitable for its performance. However, Rasch[13] pointed out deficiencies in Barnes' methodology and concluded from an analysis of late 17th and early 18th century music theory that equal temperament is a ''particularly appropriate tuning'' for the ''Well-Tempered Clavier.''

D. Equal temperament

Equal temperament, defined by equally spaced half tones, gradually took over as the standard, replacing the well temperaments during the 19th Century. The frequency ratio between any two adjacent notes is $2^{1/12}$:1. The frequency ratios used to construct a 12-tone scale in equal temperament are shown in Table I, column (f). These values remain the same when the notes are modulated to other keys.

Although equal temperament appeared in references as far back as the 16th Century, it was not adopted sooner in part because it was difficult to tune keyboard instruments to equal temperament. There continues to be a school of thought that equal temperament is a poor compromise (see Doty[12]), since it does not use any ratio with small integers except for the octave. However, this study found that equal temperament performs strongly in approximating perfect harmony when all keys and all consonant intervals are given equal weighting compared to the particular cases of the other tunings examined and also performs well for the three ''Well-Tempered Clavier'' preludes examined.

III. GENERAL EVALUATION OF FOUR TUNINGS

A. Data

The frequency ratios given for each of the tunings in Table I and for perfect harmony in the introduction were converted into cents. One cent is defined as 1/1200 of an octave and, therefore, a ratio, r, is equivalent to x cents, where $x=1200\log_2 r$. The deviation between perfect harmony and each tuning is shown in Table II, with the absolute deviation being the absolute value of the data given in that table. Because each scale consists of 12 fixed pitch classes, no distinction is made between enharmonically equivalent intervals, and keys are referred to by the labels given in Table II.

Just noticeable differences in frequency are explained by Hall[19] based on the work by Wier, Jesteadt, and Green[20] and Harris[21] that characterized the sensitivity to changes in frequency for pure tones as a function of loudness and a reference frequency. For much of the frequency range of a piano, the just noticeable difference for pure tones is about 1 Hz, and for complex waveforms, it may be as little as 0.1 Hz at low frequencies. Discrimination between two tones depends on multiple factors, such as the choice of interval, absolute pitch level, tone duration, and the musical training of the listener. A rough rule of thumb states that differences of less than five cents are generally considered imperceptible. However, Hall reports that under certain conditions, such as in the case of a well-trained musician listening deliberately for mis-

TABLE II. Deviations between perfect harmony and four tunings, measured in cents. Each value represents the difference in the size of the interval created by a specific tuning (e.g., just intonation) and the interval defined as perfect harmony. For each tuning, the deviations are found for 72 intervals (six consonant intervals of interest times 12 keys).

	Key											
	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B
Just intonation (JI)												
m3	0.00	−41.06	−21.51	−41.06	0.00	0.00	−21.51	0.00	−41.06	0.00	−21.51	0.00
M3	0.00	0.00	0.00	0.00	41.06	0.00	41.06	0.00	0.00	41.06	0.00	41.06
p4	0.00	−19.55	0.00	0.00	0.00	21.51	0.00	0.00	0.00	21.51	0.00	0.00
p5	0.00	0.00	−21.51	0.00	0.00	0.00	19.55	0.00	0.00	0.00	−21.51	0.00
m6	0.00	−41.06	0.00	−41.06	0.00	0.00	0.00	0.00	−41.06	0.00	−41.06	0.00
M6	0.00	21.51	0.00	0.00	41.06	21.51	41.06	0.00	0.00	21.51	0.00	41.06
Meantone temperament (MT)												
m3	−5.38	−5.38	−5.38	−46.44	−5.38	−46.44	−5.38	−5.38	−5.38	−5.38	−46.44	−5.38
M3	0.00	41.06	0.00	0.00	0.00	0.00	41.06	0.00	41.06	0.00	0.00	41.06
p4	5.38	5.38	5.38	−35.68	5.38	5.38	5.38	5.38	5.38	5.38	5.38	5.38
p5	−5.38	−5.38	−5.38	−5.38	−5.38	−5.38	−5.38	−5.38	35.68	−5.38	−5.38	−5.38
m6	−41.06	0.00	0.00	−41.06	0.00	−41.06	0.00	0.00	0.00	0.00	−41.06	0.00
M6	5.38	46.44	5.38	5.38	5.38	5.38	46.44	5.38	46.44	5.38	5.38	5.38
Well temperament (WT)												
m3	−21.51	−15.64	−9.78	−21.51	−9.78	−21.51	−15.64	−15.64	−15.64	−3.91	−21.51	−15.64
M3	3.91	21.51	9.78	15.64	15.64	3.91	21.51	9.78	21.51	15.64	9.78	15.64
p4	0.00	0.00	5.87	0.00	0.00	0.00	5.87	5.87	0.00	5.87	0.00	0.00
p5	−5.87	0.00	−5.87	0.00	0.00	0.00	0.00	−5.87	0.00	0.00	0.00	−5.87
m6	−21.51	−15.64	−9.78	−15.64	−3.91	−21.51	−9.78	−15.64	−15.64	−3.91	−21.51	−9.78
M6	3.91	21.51	15.64	21.51	15.64	9.78	21.51	9.78	21.51	15.64	15.64	15.64
Equal temperament (ET)												
m3	−15.64	−15.64	−15.64	−15.64	−15.64	−15.64	−15.64	−15.64	−15.64	−15.64	−15.64	−15.64
M3	13.69	13.69	13.69	13.69	13.69	13.69	13.69	13.69	13.69	13.69	13.69	13.69
p4	1.96	1.96	1.96	1.96	1.96	1.96	1.96	1.96	1.96	1.96	1.96	1.96
p5	−1.96	−1.96	−1.96	−1.96	−1.96	−1.96	−1.96	−1.96	−1.96	−1.96	−1.96	−1.96
m6	−13.69	−13.69	−13.69	−13.69	−13.69	−13.69	−13.69	−13.69	−13.69	−13.69	−13.69	−13.69
M6	15.64	15.64	15.64	15.64	15.64	15.64	15.64	15.64	15.64	15.64	15.64	15.64

tuning, the ear may be much more discriminating. Perceptual studies such as performed by Vos[22] provide validation for this observation. In Vos' study, musically trained subjects were asked to rate tempered fifths and major thirds in terms of "purity," where differences in purity ratings imply discrimination between notes. For comparisons between tones that each contained 20 harmonics, the mean subjective purity ratings for temperings of ± 2 cents with a tone duration of 0.5 s were statistically significantly different from the mean ratings for zero tempering.

Beyond the issue of whether a difference is noticeable is whether it is tolerable in terms of pleasant sounding. The amount of mistuning that is tolerated depends on the function of the mistuned tone within the local tonal hierarchy and the direction of the interval in the case of melody. Deviations of 10–20 cents are common in actual performances, since wind and string instruments, for example, easily vary by such amounts. Such deviations generally do not correlate with a perception of roughness or dissonance, whereas deviations greater than 25–30 cents tend to be unpleasant sounding and difficult to use in practice. Vos[22] reported that the strong difference in subjective purity ratings between pure and tempered intervals could be mainly attributed to the occurrence of beats or roughness.

Although just intonation and meantone have a large number of zero entries in Table II, these two tunings also are susceptible to large deviations, which limit the intervals and keys that can be played. Well temperament is successful in eliminating the wolf tones for which meantone was criticized, and similarly, equal temperament is characterized by moderate, rather than extreme, deviations from pure consonance.

B. Analysis and results

1. Measures of central tendency and dispersion

A statistical summary of the absolute values of the data in Table II shows how well each tuning performs. The smaller the absolute deviations, the closer a tuning is to perfect harmony. Two measures of central tendency and three measures of variability for the absolute deviations from perfect harmony are given in Table III.

Well temperament and equal temperament have the smallest means, that is, are closest to perfect harmony on average. Just intonation has a median of zero, which is the smallest of all the tunings. In this application, the mean seems a more useful measure than the median, since it explicitly incorporates very large deviations which would tend to be unpleasant sounding.

The results in Table III for the mean absolute deviations for meantone and equal temperament agree with those presented by Rasch[7] in his Table VI as the "mean tempering of a tuning." The methodology used in this study allows as well

TABLE III. Measures of central tendency and dispersion for absolute deviations from perfect harmony. Two measures of central tendency, the mean and median, and three measures of dispersion, the variance, range, and interquartile range, are shown. Equal temperament performs as well as or better than the other tunings for three of the five parameters.

Measures	Absolute deviations			
	JI	MT	WT	ET
Mean	11.51	12.41	10.43	10.43
Median	0	5.38	9.78	13.69
Variance	270.09	264.42	60.74	36.53
Range	41.06	46.44	21.51	13.69
Interquartile range	21.51	0	11.73	13.69

for the calculation of this parameter for irregular tunings, including just intonation and well temperament.

Of greater interest are the measures of variability, which show how far each tuning deviates from perfect harmony. The variance and range of the absolute deviations show a trend of decreasing variability across the centuries, from meantone to well temperament to equal temperament, corresponding to the oldest to newest standard.[23] The interquartile range, which measures the difference between the 25th and 75th percentiles, shows the opposite trend, reflecting that the middle of the data for well temperament and equal temperament is more disperse. The zero interquartile range for meantone reflects that over half the data values were equal to 5.38, the value of the median. Just intonation has the largest interquartile range.

A graphical representation in a box-whisker plot, shown in Fig. 1, illustrates the wider variability for just intonation and meantone temperament. The ends of the ''box'' correspond to the 25th and 75th percentiles, with a line drawn at the 50th percentile, i.e., the median. The ''whiskers'' or lines at the ends of the figure correspond to the 2.5th and 97.5th percentiles, thereby capturing 95% of the data between those values. The graphs show how compact well temperament and equal temperament are overall, even though they have

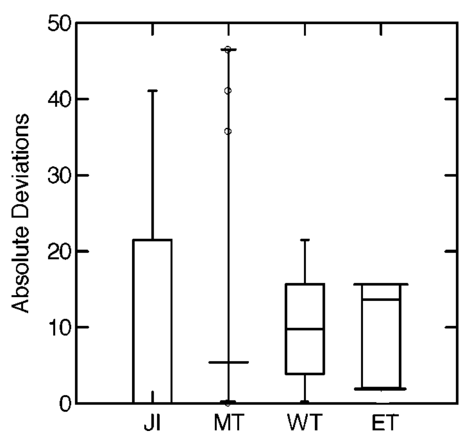

FIG. 1. Box-whisker plots of the absolute deviations of the tunings from perfect harmony. Each plot depicts the 2.5th, 25th, 50th, 75th, and 97.5th percentiles for one of the tunings, just intonation (JI), meantone temperament (MT), well temperament (WT), and equal temperament (ET). Only meantone has extreme points, designated by circles. Well temperament and equal temperament have less overall spread than just intonation and meantone temperament.

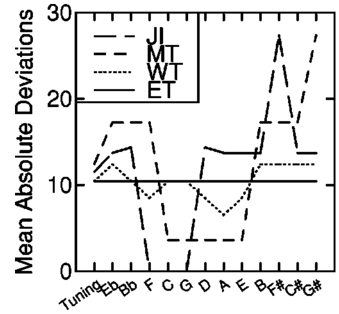

FIG. 2. Mean absolute deviations by key. The mean absolute deviation between each tuning and perfect harmony by key, as well as for the overall tuning, is displayed for each tuning. Just intonation and meantone perform the best for C and nearby keys, whereas equal temperament and well temperament exhibit less overall variability and better performance for distant keys.

large interquartile ranges. The plot also shows extreme points, values that are more than 3.0 times the interquartile range from the 25th and 75th percentiles. Only meantone temperament has such extreme points.

Another view of the variability of each tuning is provided by an examination of the mean absolute deviations from perfect harmony by key. The results are subject to the selection of C as the starting note for each tuning. As shown in Fig. 2, just intonation and meantone are close to perfect harmony for C and nearby keys, arranged in order based on fifths, but have the most extreme deviations for distant keys. On the other hand, equal temperament does not vary by key, and well temperament exhibits modest variation. These patterns of variability between keys underlie the results displayed in the box-whisker plots of Fig. 1.

Of the five measures considered in this section, two of them—the mean and variance—are explicit functions of all deviations. These seem the best suited of the measures considered here to summarize the performance of the tunings since they account for large deviations, which tend to be unpleasant sounding. For both these measures, equal temperament is as good as or better than the other tunings, with well temperament also performing strongly.

2. Distance measures

The second stage of the analysis is based on calculating the distance between each of the four tunings and perfect harmony. Six different distance matrices, Euclidean, Chebyshev, City Block, Canberra, Minkowski, and Correlation distances, are used to determine whether the results will be invariant to the choice of metric. Each of the distance metrics is an explicit function of all deviations, including the largest ones. The definitions of the distance metrics and the results are provided in Table IV.

The distance results are consistent across the six measures. Equal temperament is closer to perfect harmony than any other tuning for five of the six distance measures and tied with well temperament on the sixth measure. Well temperament also performs strongly across all six measures.

Two of the distance measures in Table IV are comparable to two other measures reported by Rasch.[7] The quadratic mean tempering of a tuning used by Rasch is a con-

TABLE IV. Distances between four tunings and perfect harmony. The names and general definitions of the distance metrics are given in the first two columns; x_{ri} is the value in cents of the ith interval of the rth tuning and x_{si} is the associated value for perfect harmony. The remaining columns show the actual distances between perfect harmony and just intonation (JI), quarter-comma meantone temperament (MT), well temperament (Werckmeister III) (WT), and equal temperament (ET), respectively. The Canberra and Correlation distances are unitless, while the other metrics are in cents.

Measures	Definition	JI	MT	WT	ET
Euclidean distance	$\delta_{rs}=\sqrt{\sum_i (x_{ri}-x_{si})^2}$	170.27	173.57	110.46	102.27
Chebyshev distance	$\delta_{rs}=\max_i \lvert x_{ri}-x_{si}\rvert$	41.06	46.44	21.51	15.64
City block distance	$\delta_{rs}=\sum_i \lvert x_{ri}-x_{si}\rvert$	828.99	893.51	750.78	750.78
Canberra distance	$\delta_{rs}=\sum_i \frac{\lvert x_{ri}-x_{si}\rvert}{x_{ri}+x_{si}}$	0.8311	0.8882	0.7612	0.7610
Minkowski distance (with $w_i=1$ and $\lambda=3$)	$\delta_{rs}=\left(\sum_i w_i \lvert x_{ri}-x_{si}\rvert^{\lambda}\right)^{1/\lambda}$	102.71	107.77	60.32	53.55
Correlation distance	$\delta_{rs}=1-\frac{\Sigma_i(x_{ri}-\overline{x}_r)(x_{si}-\overline{x}_s)}{\sqrt{\Sigma_i(x_{ri}-\overline{x}_r)^2(x_{si}-\overline{x}_s)^2}}$	0.0021	0.0022	0.0009	0.0008

stant c times the Euclidean distance, where the constant is the number of intervals in the calculation raised to the negative one-half power $[c=(72)^{-1/2}]$. Also, the maximum tempering of a tuning equals the Chebyshev distance between each tuning and perfect harmony. The generalized distance definitions provided here simplify the calculations for irregular tunings.

Figure 3 provides a graphical representation of the data in Table IV as star plots to highlight the differences between the tunings. Each star plot consists of six spokes, each corresponding to one of the distance metrics. For a given spoke, the maximum value in Table IV for a particular distance metric is plotted at the extreme end of a spoke and all other values for that metric are plotted proportionately. The star plot for equal temperament has the smallest area, showing that it is the closest to perfect harmony on these distance measures. The star plot for well temperament also has a relatively small area compared to just intonation and meantone temperament.

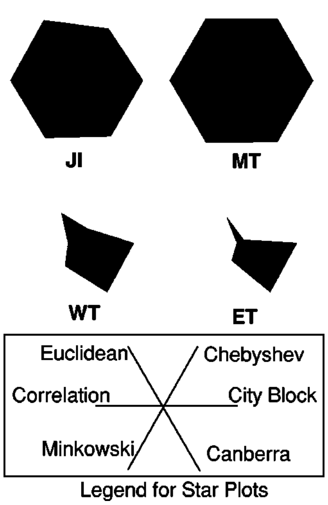

FIG. 3. Star plots of six distance measures. Six metrics, Euclidean, Chebyshev, City Block, Canberra, Minkowski, and Correlation, are used to measure the distance between a tuning and perfect harmony. The largest value for a metric is used to scale the plot on the corresponding spoke of the star plot. The star plot for equal temperament has the smallest area, which means it is the closest to perfect harmony on these measures. The plot for well temperament also has a relatively small area compared to just intonation and meantone.

The distance measures were also examined by key, with the overall patterns for each tuning closely resembling that shown in Fig. 2 for the mean absolute deviations.

3. Measure of dissonance

As explained by Sethares,[24] the concept of sensory consonance implies that consonance and dissonance depend not just on the interval between tones, but also on the spectra of the tones, which correlate with timber. The roughness or tonal dissonance experienced for two simultaneously sounded complex tones can be accounted for by interfering harmonics. The relationship between a perception of ‘‘impurity’’ and interfering harmonics is consistent with the primary findings of the perceptual experiments conducted by Vos.[22]

Based on the work of Plomp and Levelt[11] for pure tones, Sethares[10] developed a model for measuring the dissonance of two complex tones with base frequencies f and αf, as follows:

$$D=D_F+D_{\alpha F}+\sum_{i=1}^{n}\sum_{j=1}^{n} d(f_i,\alpha f_j,\nu_i,\nu_j), \qquad (1)$$

where f_i and αf_i are the partials of the two tones, ν_i are the amplitudes of the partials, and D_F and $D_{\alpha F}$ are the dissonances for each tone calculated from its own partials, with

$$D_F=1/2\sum_{i=1}^{n}\sum_{j=1}^{n} d(f_i,f_j,\nu_i,\nu_j), \qquad (2)$$

$$d(f_i,f_j,\nu_i,\nu_j)=\nu_i\nu_j(e^{-3.5s(f_i-f_j)}-e^{5.75s(f_i-f_j)}), \qquad (3)$$

TABLE V. Dissonance ratios based on a model of sensory consonance. Two different sets of amplitudes were used for the harmonic partials of the two complex tones in an interval. These were incorporated into Eq. (1) to obtain an absolute dissonance measure for each tuning, summed across the 72 intervals of Table II. Each dissonance measure was then normalized by forming the ratio of the absolute dissonance to the corresponding dissonance level for perfect harmony.

	Dissonance ratios			
Amplitudes	JI	MT	WT	ET
Based on 0.88	1.143	1.179	1.164	1.158
Based on $1/n$	1.086	1.108	1.087	1.081

$$s=\frac{0.24}{0.021f_i+19}, \tag{4}$$

and $f_i<f_j$. Sethares created dissonance curves by letting α vary continuously over an interval such as $1\leqslant\alpha\leqslant 2$. In this analysis, α assumes the values associated with the consonant intervals in Table I. Then a measure of dissonance for the 72 consonant intervals in Table II is found by summation and can be compared to the mean absolute deviation for the same 72 intervals. Since the measure of dissonance, D, depends on actual frequencies, not just the ratio of frequencies, its values will change for notes within different octaves. The calculations are performed by setting C equal to a frequency of 261.63 Hz. Two sets of amplitudes are used in the analysis: (1) following the analysis of Sethares,[10] amplitudes are set to fall at a rate of 0.88, and (2) as suggested by Doty[12] as typical of many common types of musical tones, amplitudes are decreased in direct proportion to their ordinal number (for example, the second harmonic has one-half the amplitude of the first). In both cases, $n=6$ harmonic partials are used, similar to the analyses of Sethares.

A comparable measure of dissonance is also found for the perfect harmony intervals, and then the results for each tuning are calculated as a ratio of the dissonance for the tuning compared to the dissonance for perfect harmony. With this dissonance ratio measure, 1.0 equals perfect harmony and values closer to 1.0 indicate a tuning is closer to perfect harmony.

The results for the four tunings, presented in Table V, indicate the strongest performances are associated with just intonation and equal temperament and the weakest with meantone temperament. Compared to the earlier analysis based strictly on frequencies, rather than the spectra, just intonation is viewed as more consonant and well temperament as less so. However, overall there is little difference between the four tunings due to the added complexity introduced by the interfering harmonics.

To examine the differences by key, Fig. 4 displays the dissonance ratio by key with the amplitudes based on $1/n$. As before, the results for the measure of dissonance are subject to the selection of C as the starting note for each tuning. The general pattern by key is similar for the other set of amplitudes considered. Many similarities between Figs. 2 and 4 are apparent. For example, just intonation and mean temperament again perform best for the key of C and nearby keys and reach their most extreme values for distant keys.

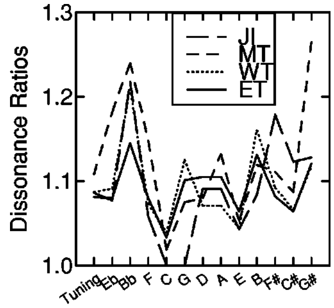

FIG. 4. Dissonance ratios by key. The dissonance ratio measure for each tuning compared to perfect harmony is displayed by key, as well as for the overall tuning. Just intonation and meantone perform the best for C and nearby keys, whereas equal temperament and well temperament exhibit less overall variability and better performance for distant keys. The performance pattern is similar to that of Fig. 2, indicating that the mean absolute deviation based on frequencies is in reasonable agreement with the dissonance ratio based on the spectra analyzed.

Similarly, equal temperament and meantone exhibit less variability than the other two tunings. However, they show considerably more variability than in Fig. 2 because of the dependence of the dissonance measure on the absolute frequencies. Equal temperament, in particular, has a variable dissonance ratio, unlike its mean absolute deviation, since the dissonance ratio is a function of the frequencies of the tones in an interval and the beats measured by their difference. Also, although in general the measure of dissonance for a given tuning decreases as the frequency increases, this is not the trend for the ratio of the dissonance of a tuning to that of perfect harmony. Nonetheless, the similarity between Figs. 2 and 4 indicates that the simple measure of mean absolute deviation provides a reasonable approximation to the dissonance ratio based on a perceptual model. Several of the distance measures discussed earlier also provide a close approximation, but the strongest relationship based on correlation measures exists between the dissonance ratio and the mean absolute deviation.

In this section, the four historical tunings have been compared to perfect harmony by means of a dissonance model derived from the results of perceptual experiments conducted by Plomp and Levelt. In this way, the dissonance measure provides an assessment of the tunings that more closely reflects human perception of consonance and dissonance and in particular accounts for the roughness associated with interfering harmonics. The later perceptual experiments conducted by Vos[22] found that the primary factor that differentiated the ratings of pure and tempered intervals was the presence of beats or roughness and that deletion of harmonic interference resulted in higher ratings. Although his research asked subjects to judge purity of tempered intervals, Vos noted that it is reasonable to assume that purity and consonance are comparable concepts. Additionally, an examination of the intervals in this study that were also examined by Vos (namely the fifths and major thirds corresponding to the regular tunings of equal temperament and meantone, as well as perfect harmony) indicates a similarity of results. The correlation coefficient between the measures of dissonance and the subjective impurity ratings for these intervals, as reported in Vos,[25] is 0.83. As tempering increases, the intervals have

TABLE VI. Weights for the consonant intervals appearing in three Bach preludes. For each piece of music, the number of beats for each of the consonant intervals was counted and classified by root note. The counts of beats were converted into weights by scaling each count by the total number of beats for a piece of music. The cells with a value of zero correspond to intervals that do not occur in the music.

	Key												
Prelude 1, C Major													
	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B	Total
m3	0.004	0	0.048	0	0.060	0.015	0.004	0.019	0	0.023	0	0.047	0.220
M3	0.130	0	0.008	0	0	0.035	0	0.035	0.004	0	0	0	0.214
p4	0.017	0	0.033	0	0.029	0	0	0.033	0	0.013	0	0.005	0.130
p5	0.079	0	0.027	0	0	0.021	0	0.081	0	0.004	0	0	0.213
m6	0	0	0	0	0.050	0	0	0.015	0	0	0	0.023	0.088
M6	0.029	0	0.013	0.004	0	0.021	0	0.042	0.019	0.004	0.004	0	0.136
Total	0.260	0	0.129	0.004	0.140	0.092	0.004	0.225	0.023	0.044	0.004	0.075	1.000
Prelude 3, C# Major													
	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B	Total
m3	0.023	0.002	0.007	0.058	0	0.106	0.005	0	0.005	0	0.016	0	0.222
M3	0	0.062	0	0.005	0.005	0	0.032	0	0.039	0.005	0	0.005	0.152
p4	0.005	0.018	0	0.044	0	0.005	0.005	0	0.079	0	0.014	0	0.169
p5	0.002	0.088	0	0.012	0	0	0.009	0	0.025	0	0.009	0.005	0.150
m6	0.035	0	0.005	0.007	0	0.072	0	0	0	0	0.023	0	0.141
M6	0.002	0.044	0	0.030	0.002	0	0.030	0	0.039	0	0.002	0.016	0.166
Total	0.067	0.215	0.012	0.155	0.007	0.182	0.081	0	0.187	0.005	0.065	0.025	1.000
Prelude 5, D Major													
	C	C#	D	Eb	E	F	F#	G	G#	A	Bb	B	Total
m3	0	0.018	0.035	0.003	0.027	0	0.083	0.012	0.038	0.005	0	0.029	0.249
M3	0.002	0	0.065	0	0	0.002	0.002	0.011	0	0.051	0.002	0.002	0.134
p4	0	0	0.002	0	0.027	0	0.002	0	0	0.119	0	0.002	0.151
p5	0	0	0.066	0	0.006	0	0.006	0.011	0	0.080	0.002	0.003	0.174
m6	0	0.008	0.002	0.002	0	0	0.060	0	0.009	0.041	0.002	0.002	0.124
M6	0.002	0	0.042	0	0.018	0.026	0	0.023	0	0.044	0.003	0.011	0.168
Total	0.003	0.026	0.211	0.005	0.079	0.027	0.153	0.056	0.047	0.340	0.008	0.047	1.000

larger dissonance measures, with major thirds having larger dissonance measures than fifths. However, the major thirds are less sensitive to large temperings than are the fifths, which is consistent with how equal temperament treats these two intervals. The dissonance measures for the fifths tend to follow an exponential curve, as modeled by Vos, both when examined across a tuning and for each key separately. Future work with perceptual experiments that incorporate more than regular tunings and additional intervals would be useful for comparison with modeled results.

IV. SPECIFIC EVALUATION OF TUNINGS BASED ON THREE PRELUDES

The analysis presented in Sec. III provides a general characterization of the performance of the four tunings using a variety of quantitative measures. Hall[2] argued that objective measurement of how well a tuning performs should be calculated with respect to specific pieces of music. Since a given piece will not likely include all consonant intervals in equal proportion, the performance of a tuning will vary from that described in the general case. This section presents an analysis for three preludes from Bach's ''Well-Tempered Clavier'' for which intervals are weighted in proportion to the duration of their occurrence in the music.

A. Data

Data were collected for three pieces from the ''Well-Tempered Clavier, Book I:'' Prelude 1, C Major; Prelude 3, C# Major; and Prelude 5, D Major. When the notes in a chromatic scale are arranged in order based on fifths, they stand as: Eb Bb F C G D A E B F# C# G#. Therefore, even though C, C#, and D are adjacent tones in a single scale, C and D are central keys, while C# is a peripheral key.

For each piece of music, the number of beats for each of the consonant intervals (m3, M3, p4, p5, m6, and M6) was counted and classified by root note. For example, in Prelude 1, perfect fifths occur for a duration of 51 equivalent beats, 19 of which have C as the root note. Chords were separated into their component two-note intervals. The counts of beats were converted into weights by scaling each count by the total number of beats for a piece of music. The resulting weights for the three preludes are given in Table VI. The cells with a value of zero correspond to intervals that do not occur in the music. This method of weights differs from that of Hall[2] in that the duration, not just the occurrence, of an interval is used to determine the weight.

A review of the pattern of zeros, as well as the rows and columns labeled ''Total'' in Table VI, indicates that all intervals do not appear in approximately equal proportion. For example, in Prelude 3, only 1.2% of all consonant intervals have D as a root note, compared to 21.5% for the tonic C#. Although Barnes[9] argues that intervals should occur more or less often according to their tuning quality, this seems too restrictive, as a piece written for equal temperament should not be expected to have intervals based on all notes appear in equal proportion. Charts of duration versus the absolute deviations from perfect harmony do show that intervals with large absolute deviations tend to be correlated with small

TABLE VII. Weighted measures of central tendency, dispersion, and distance for the absolute deviations from perfect harmony for three Bach preludes and the overall tuning. The weights in Table VI are applied to the absolute deviations to obtain weighted measures for the preludes. Equal weights of 1/72 are used for the overall tuning. Overall, just intonation and meantone temperament exhibit the most variability between different pieces of music, whereas well temperament and equal temperament are more stable and more moderate in size of deviations.

		Absolute deviations			
Measures	Tuning/prelude	JI	MT	WT	ET
Weighted mean	C Major prelude	2.51	5.30	7.57	10.36
	C# Major prelude	10.16	29.23	14.21	10.71
	D Major prelude	14.81	4.36	9.73	10.69
	Tuning	11.51	12.41	10.43	10.43
Weighted variance	C Major prelude	128.75	120.73	33.81	37.49
	C# Major prelude	268.09	622.53	110.15	36.54
	D Major prelude	239.56	83.85	30.50	37.39
	Tuning	270.09	264.42	60.74	36.53
Weighted Euclidean distance	C Major prelude	7.35	9.91	9.11	12.03
	C# Major prelude	19.22	34.55	17.25	12.29
	D Major prelude	21.17	6.17	11.17	12.31
	Tuning	20.07	20.46	13.02	12.05
Weighted Chebyshev distance	C Major prelude	21.51	46.44	21.51	15.64
	C# Major prelude	41.06	46.44	21.51	15.64
	D Major prelude	41.06	46.44	21.51	15.64
	Tuning	41.06	46.44	21.51	15.64

durations; however, some intervals with small or zero absolute deviations are also associated with small durations.

Finally, the weights in Table VI were combined with the absolute deviations derived from Table II to create performance measurements discussed in the next section. One limitation of this analysis is the need to assume a particular key as the starting point for each tuning, in this case the key of C. This choice of disposition necessarily impacts the results. Therefore, the results reported here are illustrative and would not necessarily be the same if another disposition were selected.

B. Analysis and results

1. Measures of central tendency and dispersion

Two measures of central tendency and dispersion for the weighted absolute deviations between each tuning and perfect harmony for the three Bach preludes are included in Table VII. Because of the similarity of results between measures, only a subset of the measures discussed in Sec. III appears in this and the following section. The ones selected are explicit functions of all deviations. Table VII also includes the values for the overall tuning for comparison, with each interval given equal weight in the calculations.

The weighted mean of the absolute deviations demonstrates the variability in meantone temperament between pieces. Although meantone performs well for the C major and D major preludes, the frequent occurrence in the C# prelude of minor and major thirds and sixths with large absolute deviations from perfect harmony contribute to the large mean for that piece. Meantone temperament has not been proposed as a likely tuning for the ''Well-Tempered Clavier,'' and the weighted mean absolute deviation supports that position. Just intonation, as defined in Table I, performs well for the C major prelude, but is the worst of the four tunings for the D major prelude. Well temperament and equal temperament demonstrate strong consistency from piece to piece. Although they are not always the best tuning, and equal temperament is sometimes the worst, both perform well across the different preludes.

Although Rasch[7] does not analyze specific pieces of music, he introduces a weighting scheme based on the circle of fifths to determine the mean tempering of keys. Mean tempering is equivalent to the mean absolute deviations reported here. Based on a simplified view of the relative importance of intervals within a key, Rasch concludes that for meantone, the key of C major has the smallest mean tempering and that for the central keys of F, C, G, and D major, meantone tuning is the best tuning of the regular tunings considered. The analysis of the three preludes from the ''Well-Tempered Clavier,'' which is based on weights derived from the music, shows that meantone performed slightly better for D major than C major, but considerably worse for C#, a peripheral key. Since Rasch's results are based on a simplified assumption on weights, small differences in findings such as reported here should be expected based on individual music. Also, just intonation outperformed meantone in the key of C, indicating how an irregular tuning may be closer to perfect harmony for some pieces.

Table VII also presents a measure of dispersion, the weighted variance for the absolute deviations from perfect harmony for the three preludes and overall tuning. Meantone temperament performs poorly for the C# major prelude, and just intonation demonstrates considerably more variability than equal temperament. Both well temperament and equal temperament perform the best on this measure across the preludes. The results in this section depend on the initial selection of C as the starting note for each tuning and are applicable to the particular version of just intonation defined in Table I.

2. Distance measures

Two distance measures for the three preludes and the overall tuning are included in Table VII. The values for the Euclidean distance for the overall tuning differ from the results in Sec. III by the constant $c=(72)^{-1/2}$ and equal the root mean square reported by Rasch,[7] while the Chebyshev distance equals his maximum tempering of a tuning. Additionally, the Euclidean distance metric used for the preludes equals the square root of the goodness-of-fit measure used by Hall.[2]

The weighted Euclidean distance between each tuning and perfect harmony shows similar patterns to those found for the weighted mean and variance. All four tunings perform well for the C major prelude, while well temperament and equal temperament continue to perform best across the pieces. The pattern for the Chebyshev distance differs from that of the other statistics. The maximum absolute deviation differs from that of the overall tuning only if intervals associated with the maximum value never appear in the piece of music. This occurs only once—for the C major prelude. In this case, the minor and major thirds and sixths with an absolute deviation from perfect harmony of 41.06 for just intonation do not appear.

The analysis of the three preludes has shown that there can be considerable variability in the measures of central tendency, dispersion, and distance for a given tuning across different pieces, as well as between tunings for a single-piece. Perceptual experiments have also found this to be the case when musically trained subjects rate performances of different musical pieces, such as in the experiments conducted by Vos.[25] In that study, subjects were presented with musical fragments according to one of seven regular tunings (including equal temperament, meantone temperament, and perfect harmony) and asked to rate their acceptability. Vos determined through multiple regression that acceptability is highly correlated with the subjective purity ratings of isolated fifths and major thirds, suggesting that the acceptability ratings reflected a judgment of consonance. An analysis of variance of the experimental data revealed that mean acceptability ratings differ as a function of the tuning, the musical fragment, and the individual subjects, as well as interactions between them. When averaged across fragments and subjects, the mean acceptability ratings were about the same for tunings with between 0 and -5.4 cents deviations from perfect harmony for fifths (which includes equal temperament, meantone, and perfect harmony). However, the main effect of musical fragments was related to the mean absolute tempering, and a strong correlation was found between mean absolute tempering and overall acceptability. These results suggest that the mean absolute deviations (which are equivalent to mean absolute tempering) provide a reasonable approximation to perceptual ratings of consonance.

V. CONCLUSION

The general evaluation of the four historical tunings using measures of central tendency, measures of dispersion, distance measures, and a dissonance ratio support the conclusion that equal temperament performs strongly in approximating perfect harmony when all keys and all consonant intervals are given equal weighting compared to the particular cases of 5-limit just intonation, quarter-comma meantone temperament, and well temperament (Werckmeister III) analyzed.

Additionally, both equal temperament and well temperament provide the most consistently strong performances as tunings for three preludes from Bach's "Well-Tempered Clavier." The findings here agree with the results of Rasch[13] who demonstrated that equal temperament is an appropriate tuning for the "Well-Tempered Clavier" based on evidence from German music theory. The analysis here is not intended to justify any specific temperament as the one Bach intended for this piece, but rather to use quantitative measurements of performance compared to perfect harmony to judge the suitability of these tunings. The results for the specific preludes are subject to the selection of C as the starting note for each tuning.

The measures of central tendency, dispersion, distance, and dissonance used in both the general and composition-specific analyses can be readily applied to other tunings and other pieces of music to evaluate both relative and absolute performance. These measures include ones that are equivalent to those defined by Rasch[7] for regular tunings and by Hall[2] for specific pieces of music. As presented here, these measures can be readily interpreted as measuring centrality and variability of a tuning compared to the standard of perfect harmony, as well as overall distance from pure consonance. The mean absolute deviation appears to be the measure best correlated with the dissonance ratio that reflects sensory dissonance, as well as with the results of perceptual studies of subjective acceptability. Comparing the mean absolute deviation with the results of perceptual experiments for both regular and irregular tunings should be considered for further research.

ACKNOWLEDGMENTS

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[15] The ratios, when multiplied times an arbitrary frequency selected for the tonic, produce the frequencies for the notes in the scale. Because consonance is defined by frequency ratios, rather than absolute frequencies, Table I focuses on the intervals relative to the tonic associated with each tuning.
[16] Although proponents of just intonation can create a large number of pure tones by means of programmable synthesizers, this study uses only the 12 tones defined for just intonation by Table I in order to compare 12-note tuning methods.
[17] The term comma refers to a small interval that usually arises due to two different methods of determining a tone or interval. The syntonic comma is the difference between a major third in Pythagorean tuning, which has a ratio of 81/64, and a major third in 5-limit just intonation, which has a ratio of 5/4. This difference is the interval 81/80. The difference between

two ratios is calculated as one ratio divided by the other. In this case, (81/64)/(5/4) yields $(81/64)\times(64/80)=81/80$.

[18] The Pythagorean comma is calculated as $(3/2)^{12}/2^{7}=531441/524288$.

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